2.1

The Diffusion Equation

Contents

1 Introduction

This project is programmed in **MATLAB R2014a**. Consult section A for program documentation, listings and information on the structure of the programming for the project. This report is written in LaTeX 2_{ε}.

2 Analytic Solutions

Q1. The function $F(x,t) \equiv \frac{\theta(x,t)-\theta_0}{\theta_1-\theta_0}$ satisfies the diffusion equation on $(x,t) \in (0,\infty)^2$:

$$F_t(x,t) = \frac{\theta_t(x,t)}{\theta_1-\theta_0} = \alpha\frac{\theta_{xx}(x,t)}{\theta_1-\theta_0} = \alpha F_{xx}(x,t)$$

The initial and boundary conditions on θ are equivalent to conditions on F: $\forall x \in (0,\infty)$, $\forall t > 0$

$$\theta(x,0) = \theta_0 \iff F(x,0) = 0$$

$$\theta(0,t) = \theta_1 \iff F(0,t) = 1$$

$$\lim_{x\to\infty} \theta(x,t) = \theta_0 \iff \lim_{x\to\infty} F(x,t) = 0$$

$$\lim_{x\to\infty} \theta_x(x,t) = 0 \iff \lim_{x\to\infty} F_x(x,t) = 0$$

Hence, the diffusion equation can be solved for F, in a form that is independent of θ_0 and θ_1. $F(x,t)$ depends on the physical quantities x, t, α of dimension L, T, L^2T^{-1} respectively. As a consequence of Bridgman's theorem[1], as F is dimensionless (being a ratio of temperatures), it can be expressed as a function of any basis of dimensionless parameters of the system, whose kernel consists of the solutions to

$$[x]^a[t]^b[\alpha]^c = L^{a+2c}T^{b-c} = 1$$

The corresponding linear system $a + 2c = b - c = 0$ has kernel spanned by $(a,b,c) = \left(1, \frac{1}{2}, \frac{1}{2}\right)$, so the dimensionless parameters have basis $\xi = xt^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}$. Thus, $\exists f : F(x,t) \equiv f(\xi)$. Substituting into the diffusion equation,

$$\frac{\partial}{\partial t}f(\xi) = \alpha\frac{\partial^2}{\partial x^2}f(\xi) \implies f'(\xi)\frac{\partial \xi}{\partial t} = \alpha\frac{\partial}{\partial x}\left(f'(\xi)\frac{\partial \xi}{\partial x}\right) = \alpha f'(\xi)\frac{\partial^2\xi}{\partial x^2} + \alpha f''(\xi)\left(\frac{\partial \xi}{\partial x}\right)^2$$

Since $\frac{\partial \xi}{\partial t} = -\frac{x}{2\sqrt{\alpha t^3}}$, $\frac{\partial \xi}{\partial x} = \frac{1}{\sqrt{\alpha t}}$, and $\frac{\partial^2 \xi}{\partial x^2} = 0$,

$$-\frac{f'(\xi)}{2\sqrt{\alpha t^3}} = \frac{\alpha f''(\xi)}{\alpha t} \implies f''(\xi) + \frac{1}{2}\xi f'(\xi) = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\left(\frac{\xi}{2}\right)^2} f'(\xi)\right) = 0 \implies f'(\xi) = Ae^{-\left(\frac{\xi}{2}\right)^2}$$

$$\implies f(\xi) = A\int_0^{\xi} e^{-\left(\frac{t}{2}\right)^2}\mathrm{d}t + B = 2A\int_0^{\frac{\xi}{2}} e^{-u^2}\mathrm{d}u + B = A'\int_{\frac{\xi}{2}}^{\infty} e^{-u^2}\mathrm{d}u + B'$$

The initial and boundary conditions yield the following:
$F(x,0) = 0 \implies f(\infty) = 0$, so $B' = 0$. $F(0,t) = 1 \implies f(0) = 1$, so $A' = \frac{2}{\sqrt{\pi}}$.

Note that either of the unused boundary conditions is consistent: $F(\infty,t) = 0 \implies f(\infty) = 0$ and $F_x(\infty,t) = 0 \implies \lim_{\xi\to\infty}\frac{f'(\xi)}{\sqrt{\alpha t}} = \lim_{\xi\to\infty}\frac{A}{\sqrt{\alpha t}}e^{-\left(\frac{\xi}{2}\right)^2} = 0 = 0$ (a tautology).

Thus,

$$f(\xi) = \frac{2}{\sqrt{\pi}}\int_{\frac{\xi}{2}}^{\infty} e^{-u^2}\mathrm{d}u$$

[1] See, for example, http://www.archim.org.uk/lecturenotes/ia/dynamics.pdf (sections 1.2.3 and 1.2.5; access date: 26/4/2015); Bridgman's theorem states that any physical variable can be expressed in the form $C(\theta_i)\prod X_j$, where $\{X_j\}$ is a set of physical variables of distinct dimensions that the system depends on, and $\{\theta_i\}$ is a basis of dimensionless quantities. It essentially follows from the fact that any other relationship would involve the summation of dimensionally-distinct variables (considering, for example, Taylor expansions of functions), which is forbidden.

Q2: First Problem. Take, as an initial ansatz, a function $S(X)$ dependent only on X that solves (10)–(12) (i.e. the PDE subject to the boundary conditions). (11) and (12) then imply $S(0) = 1$ and $S(1) = 0$, whence $\dot{S}(X) \equiv 0 \equiv S''(X) \equiv AX + B$, $A, B \in \mathbb{R}$, so by the BCs, $S(X) \equiv 1 - X$ is a solution to this problem.[2]

Hence, a function $U(X,T)$ solves the original problem ((10)–(13)) iff the function $V(X,T) \equiv U(X,T) - S(X)$ solves the following, transformed problem $\forall X \in (0,1)$, $\forall T \in (0,\infty)$

$$V_T(X,T) \equiv V_{XX}(X,T) \tag{O}$$

$$V(X,0) \equiv X - 1 \tag{A}$$

$$V(0,T) \equiv 0 \tag{B}$$

$$V(1,T) \equiv 0 \tag{C}$$

Hence, we can proceed by finding a solution to this problem. Suppose, as a second ansatz, a non-zero function $V(X,T) \equiv \xi(X)\tau(T)$ satisfying (O) subject to BCs (B) and (C), which imply $\xi(0) = \xi(1) = 0$. Then $-\frac{\xi''(X)}{\xi(X)} \equiv -\frac{\dot{\tau}(T)}{\tau(T)} \equiv \lambda \in \mathbb{R}$.

Solving firstly for ξ,

$$\xi''(X) + \lambda\xi(X) \equiv 0 \tag{1}$$

Suppose $\lambda \leq 0$. Then $\xi(X) \equiv Ae^{\sqrt{-\lambda}X} + Be^{-\sqrt{-\lambda}X}$. $\xi(0) = A + B = 0$, so $\xi(1) = A\left(e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}}\right) = 0$, so $A = B = 0$ (a contradiction). Similarly, $\lambda = 0 \implies \xi(X) \equiv AX + B \implies B = 0 \wedge A + B = 0 \implies A = B = 0$ (a contradiction). Thus, $\lambda \geq 0$, whence

$$\xi(X) \equiv A\sin(\sqrt{\lambda}X) + B\cos(\sqrt{\lambda}X)$$

$\xi(0) = B = 0$, so $\xi(1) = A\sin(\sqrt{\lambda}) = 0$, so $\lambda = n^2\pi^2$, $n \in \mathbb{N}$, so

$$\xi(X) \equiv A\sin(n\pi X), \ n \in \mathbb{N}$$

Solving secondly for τ, $\dot{\tau}(T) + \lambda\tau(T)$, so $\tau(T) \equiv Ce^{-n^2\pi^2 T}$. Thus, $\forall n \in \mathbb{N}$, a solution to (O), (B), (C) is

$$V(X,T) \equiv De^{-n^2\pi^2 T}\sin(n\pi X)$$

By linearity of (O), (B), (C), a more general solution to this problem is

$$V(X,T) \equiv \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 T}\sin(n\pi X)$$

Imposing (A), a solution to the transformed problem can be obtained. Thus, suppose

$$V(X,0) \equiv \sum_{n=1}^{\infty} a_n \sin(n\pi X) \equiv X - 1$$

Let $n \in \mathbb{N}$. Equation (1) is the eigenvalue equation for the Sturm-Liouville operator $\mathcal{L}(\xi) = (p\xi')' = \xi''$ (with $p(x) \equiv 1$), which is self-adjoint on $[0,1]$ (as, for any two solutions ξ_1, ξ_2, $\left[p(x)\det\begin{pmatrix} \xi_1(x) & \xi_2(x) \\ \xi_1'(x) & \xi_2'(x) \end{pmatrix}\right]_0^1 = 0$). Thus, its eigenfunctions $\{\sin(n\pi X)\}_{n=1}^{\infty}$ are orthogonal, so

$$a_n \int_0^1 \sin(n\pi t)^2 \mathrm{d}t = \frac{1}{2}a_n = \int_0^1 (t-1)\sin(n\pi t)\mathrm{d}t = \frac{1}{n\pi}$$

[2] It follows from this argument that any solution must have this form, but it is easy to check that it is indeed a solution.

Thus, a solution is $V(X,T) \equiv -\sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-n^2\pi^2 T} \sin(n\pi X)$, so the original problem is solved by

$$U(X,T) \equiv 1 - X - \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-n^2\pi^2 T} \sin(n\pi X)$$

Q2: Second Problem. The second problem can be solved similarly. The same transformation (via the initial ansatz, which the BCs determine as $S(X) \equiv 1$) yields the transformed problem

$$V_T(X,T) \equiv V_{XX}(X,T)$$

$$V(X,0) \equiv -1$$
$$V(0,T) \equiv 0$$
$$V_X(1,T) \equiv 0$$

In terms of the second (separated) ansatz, the boundary conditions become $\xi(0) = \xi'(1) = 0$, whence $\lambda > 0$ (checking as before), whence $\xi(0) = B = 0$, so $\xi'(1) = A\sqrt{\lambda}\cos(\sqrt{\lambda}) = 0$, so $\lambda = \left(n - \frac{1}{2}\right)^2 \pi^2$, $n \in \mathbb{N}$, so

$$\xi(X) \equiv A \sin\left(\left(n - \frac{1}{2}\right)\pi X\right)$$

and

$$V(X,T) = e^{-\left(n-\frac{1}{2}\right)^2 \pi^2 T} \sin\left(\left(n - \frac{1}{2}\right)\pi X\right)$$

A solution to the transformed problem is hence

$$V(X,T) = \sum_{n=1}^{\infty} b_n e^{-\left(n-\frac{1}{2}\right)^2 \pi^2 T} \sin\left(\left(n - \frac{1}{2}\right)\pi X\right)$$

subject to

$$\sum_{n=1}^{\infty} b_n \sin\left(\left(n - \frac{1}{2}\right)\pi X\right) = -1$$

The new boundary conditions on the separated ansatz satisfy the self-adjointness condition (for the same Sturm-Liouville operator on $[0,1]$), so each eigenfunction is orthogonal, whence

$$b_n \int_0^1 \sin\left(\left(n - \frac{1}{2}\right)\pi t\right)^2 \mathrm{d}t = \frac{1}{2} b_n = \int_0^1 -\sin\left(\left(n - \frac{1}{2}\right)\pi t\right) \mathrm{d}t = -\frac{1}{\left(n - \frac{1}{2}\right)\pi}$$

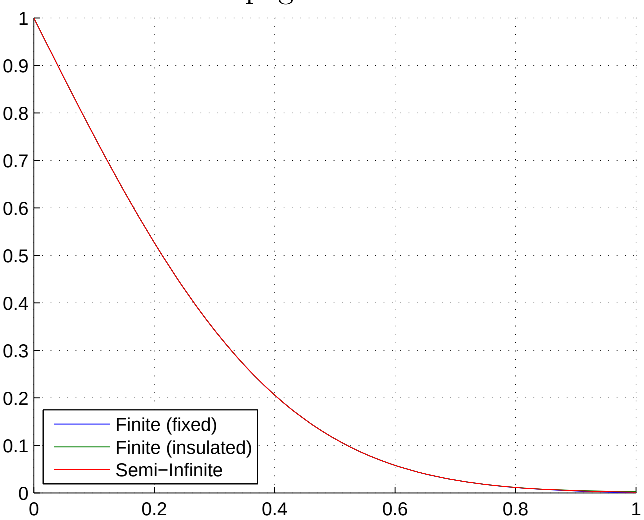
Thus, a solution to the original second problem is

$$U(X,T) \equiv 1 - \sum_{n=1}^{\infty} \frac{2}{\left(n - \frac{1}{2}\right)\pi} e^{-\left(n-\frac{1}{2}\right)^2 \pi^2 T} \sin\left(\left(n - \frac{1}{2}\right)\pi X\right)$$

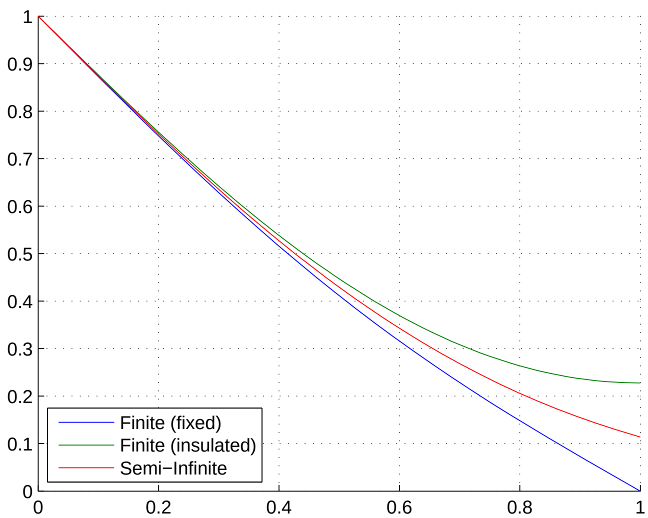
Q2: Comparisons. Here are comparisons of the three solutions to the diffusion equation (plotted on the y-axis), evaluated on $X \in [0, 1]$ (plotted on the x-axis) at times $T > 0$ using series summation or erfc, where appropriate. Analysis follows on the next page.

X	$U_1(X,T)$	$U_2(X,T)$	$F(X,T)$
0.0	1.0000000	1.0000000	1.0000000
0.1	0.8726029	0.8761313	0.8743671
0.2	0.7479073	0.7557519	0.7518296
0.3	0.6283430	0.6421696	0.6352563
0.4	0.5158250	0.5383535	0.5270893
0.5	0.4115664	0.4468241	0.4291953
0.6	0.3159643	0.3695989	0.3427817
0.7	0.2285685	0.3081944	0.2683816
0.8	0.1481328	0.2636728	0.2059032
0.9	0.0727422	0.2367138	0.1547289
1.0	-0.0000000	0.2276884	0.1138463

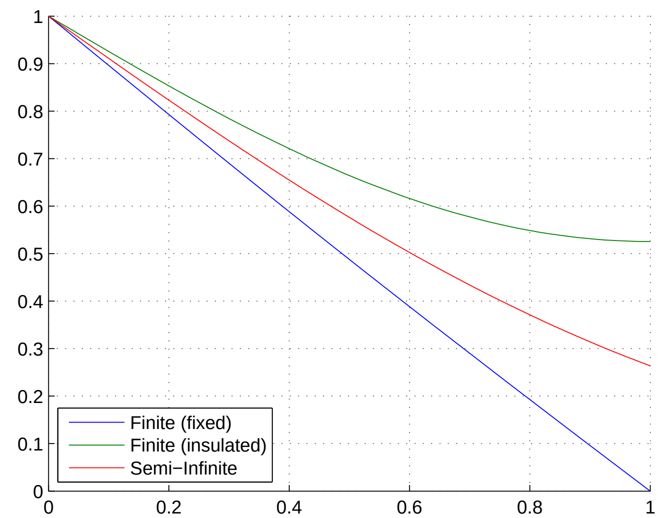
(a) $T = 0.2$



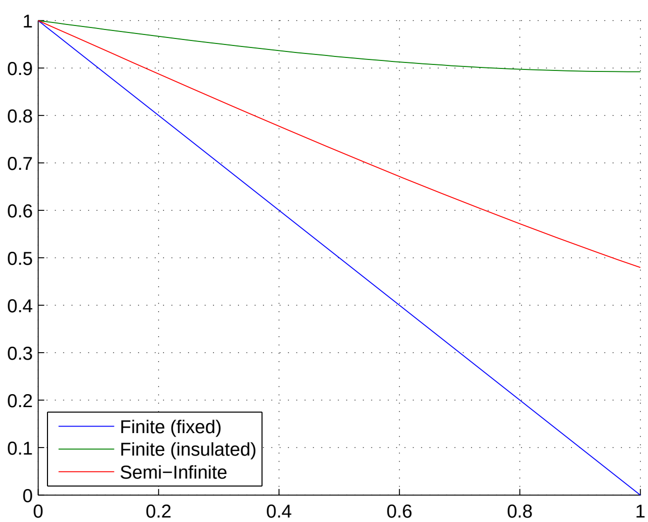
(b) $T = 0.05$



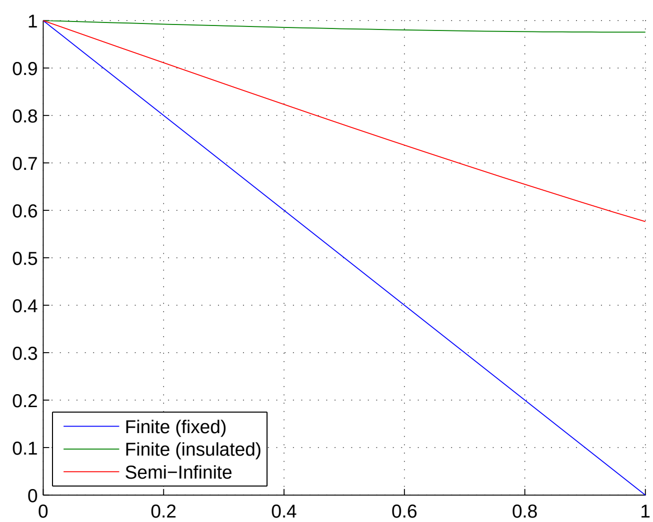
(c) $T = 0.2$



(d) $T = 0.4$



(e) $T = 1.0$



(f) $T = 1.6$

Figure 1: Solutions to the diffusion equation.

Here again are the analytic forms of the three solutions being studied, respectively semi-infinite, fixed finite and insulated finite.

$$\begin{aligned} F(X,T) =& \frac{2}{\sqrt{\pi}} \int_{\frac{X}{2\sqrt{T}}}^{\infty} e^{-u^2} \mathrm{d}u \\ U_1(X,T) =& 1 - X - \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-n^2\pi^2 T} \sin(n\pi X) \\ U_2(X,T) =& 1 - \sum_{n=1}^{\infty} \frac{2}{\left(n-\frac{1}{2}\right)\pi} e^{-\left(n-\frac{1}{2}\right)^2\pi^2 T} \sin\left(\left(n-\frac{1}{2}\right)\pi X\right) \end{aligned}$$

The number of terms summed, $n = 400$, was chosen heuristically by plotting each equation, at various values of $T \in (0,2]$, with different, exponentially-increasing values of n simultaneously to determine values that render the curves almost indistinguishable.

As an initial sanity check, it is clear that **the boundary conditions are indeed satisfied by the solutions**. At all times, each one is fixed at 1 at $X = 0$, and at $X = 1$, U_1 is fixed at 0, the slope of U_2 is 0 and F, which is defined beyond $X = 1$, appears to converge to 0 as $X \to \infty$. Likewise, the pointwise limit as $T \to 0$ appears to be 0 on $X \in (0,1)$.

Moreover, **the functions become very (uniformly) close to each other as** $T \to 0$, even at $T = 0.05$. The change in temperature w.r.t. time at each point is determined by temperatures in the neighbourhood of each point in space, which are initially zero away from $X = 0$. Each physical situation is the same near $X = 0$, so it is expected that the functions behave similarly for small times, before the influence of higher temperatures has spread far rightward.

At all times, each function is **convexly decreasing w.r.t.** X. The monotonicity mimicks the initial condition because it is preserved in time – the temperature at each point tends to average out the temperatures on either side of their neighbourhoods, since, from the definition of U_{XX},

$$U_T(X,T) = U_{XX}(X,T) = \lim_{h\to 0} \frac{2}{h^2} \left(\frac{U(X+h)+U(X-h)}{2} - U(X,T) \right) \tag{2}$$

This also means that **the temperature at each point in space is always increasing with time**, in each case. In fact, **as** $T \to \infty$**, the three solutions behave differently**:

- U_1 achieves a local averaging of temperature subject to fixed temperatures at the endpoints $X = 0, 1$, which leads to the linear steady state evidenced in later times.
- U_2 represents a system insulated at $X = 1$, where the system seeks to match the temperature immediately to the left of it. Hence, the temperature everywhere tends to match the fixed temperature at $X = 0$, namely 1.
- F similarly tends to match the fixed temperature of 1, but is nowhere insulated, so the influence of increasing temperature spreads out towards infinity. Hence, at each finite time, the temperature approaches 0 as $X \to \infty$, though there is pointwise convergence with time to 1 everywhere in space.

A final interesting feature of the graphs is that they **respect a uniform ordering** $\forall X \in [0,1] \quad \forall T \in [0,\infty) \quad U_1(X,T) \le F(X,T) \le U_2(X,T)$. Intuitively, a low fixed temperature of 0 at $X = 1$ means that U_1 loses heat more quickly than the others, while insulation at $X = 1$ means that U_2 retains more heat in the region $[0,1]$ than F, which loses heat past $X = 1$. Thus, the average temperature on the region $[0,1]$ is ordered as above, and as the functions converge towards their local averages, this ordering is respected at each point as well.

3 Numerical Solutions

Q3: Overview. An instance of the numerical scheme, implemented as `A.Y`, is tabulated below. The script `Q3` provides an interface to this function, designed to tabulate the scheme for the values of N, C, and T suggested; in each case, $\delta t = \frac{C}{N^2}$ exactly divides 0.4, which generates multiples of T. For cases where this division is inexact, the values output by the script do not correspond exactly to the suggested values of T.

Note that the condition $\forall m \geq 0 \quad U^m_{N-1} = U_{N+1}$ can be used to specify the boundary condition $U_X(1,T) \equiv 0$ because it approximates the derivative U_X as a tangent of slope 0 between the points $(1 - \delta X, T)$ and the constructed point $(1 + \delta X, T)$. This is an instance of the *method of images*, whereby a boundary condition is represented by a symmetrical extension of the function (onto the domain $[0, 1 + \delta X]$). By (2), $U(1,T)$ approximately converges with time towards the average of U^m_{N-1} and U^m_{N+1}, i.e. towards U^m_{N-1}, which gives a derivative at U^m_N of approximately zero by the mean value theorem and smoothness.

The following tables (produced by `Q3`) compare the numerical scheme with $N = 5$ and $C = \frac{1}{2}$ to the analytic solution. The rows and columns represent different values of T and X, respectively.

	0	0.2	0.4	0.6	0.8	1
0.4	1	0.8533726	0.7207522	0.6161461	0.5481968	0.5255470
0.8	1	0.9462581	0.8976481	0.8593019	0.8343911	0.8260876
1.2	1	0.9803012	0.9624835	0.9484279	0.9392970	0.9362534
1.6	1	0.9927795	0.9862485	0.9810965	0.9777497	0.9766340
2.0	1	0.9973534	0.9949595	0.9930710	0.9918443	0.9914353

Table 1: The numerical solutions.

	0	0.2	0.4	0.6	0.8	1
0.4	1	0.8533095	0.7210126	0.6160657	0.5487142	0.5255125
0.8	1	0.9453450	0.8960401	0.8569115	0.8317894	0.8231329
1.2	1	0.9796297	0.9612533	0.9466698	0.9373066	0.9340802
1.6	1	0.9924078	0.9855588	0.9801234	0.9766337	0.9754312
2.0	1	0.9971703	0.9946177	0.9925918	0.9912912	0.9908430

Table 2: The corresponding analytical solutions.

	0	0.2	0.4	0.6	0.8	1
0.4	0	-0.0000631	0.0002604	-0.0000804	0.0005174	-0.0000345
0.8	0	-0.0009131	-0.0016080	-0.0023904	-0.0026018	-0.0029547
1.2	0	-0.0006715	-0.0012302	-0.0017581	-0.0019904	-0.0021732
1.6	0	-0.0003717	-0.0006897	-0.0009731	-0.0011160	-0.0012029
2.0	0	-0.0001830	-0.0003418	-0.0004792	-0.0005531	-0.0005923

Table 3: The corresponding errors.

Q3: Empirical stability and accuracy. Denote the numerical approximation to U_2 as Y, with error E. For the remainder of the project, three scripts `Q3a`, `Q3b`, `Q3c` were used with parameters varied between runs.[3]

As a starting point, one can plot the absolute error[4] of Y for different values of N and C. Plotting against X (with `Q3a`) for simplicity, it is immediately clear that the Courant number $C = \frac{2}{3}$ gives an unstable scheme, with errors growing far beyond analytical values (see figure 2).

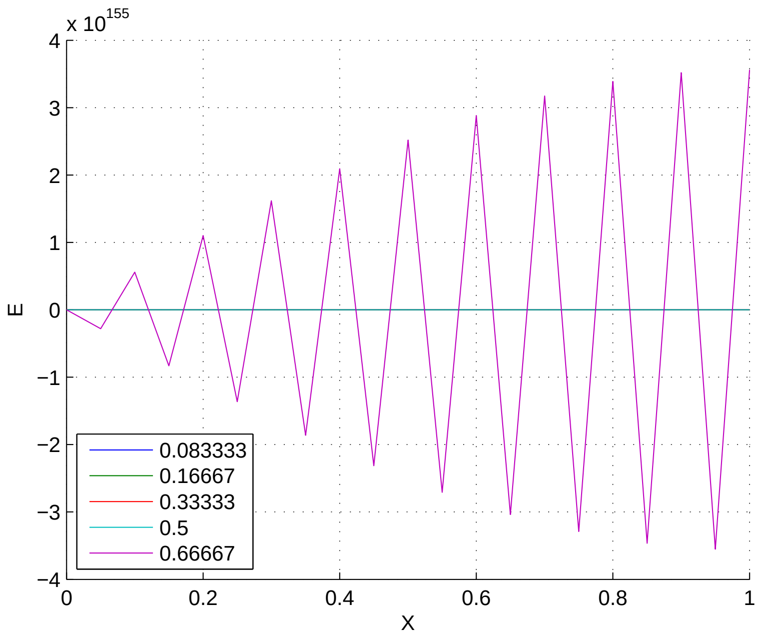


Figure 2: Instability of $C = \frac{2}{3}$; various values of C with $N = 20$, $T = 1.2$.

This behaviour is typical for different fixed settings of N and T, so the case $C = \frac{2}{3}$ may be discarded. Comparing the remaining cases, the numerical scheme is manifestly stable, with errors smaller than 3×10^3 uniformly across X in the worst of the suggested cases (namely $C = \frac{1}{2}$, $N = 5$, $T = 0/08$). Qualitatively, however, $T = 0.4$ differs from the remaining values of T (see figure 3a).

Indeed, excluding $T = 0.4$, the curves for fixed Courant numbers at fixed N, T are very similar in shape (see figure 3b for an example), with the clear trend that increasing N or T leads to a decrease in the maximum error across X.[5] This suggests that it is worth distinguishing only between Courant numbers in the following analyses of the order of accuracy of the scheme (and fixing the remaining variables).

Hence, we may consider the order of accuracy when varying X, as suggested by the improving accuracy with increasing N (i.e. decreasing δX) in the previous study. Thus, we plot, logarithm against logarithm, the absolute value of the absolute error against δX (with `Q3b`). In fact, with the case $T = 0.4$ excluded, evidence uniformly indicates (fixing various values of (X, T)) the order of accuracy of the scheme. For a unanimously typical example, see figure 4a.

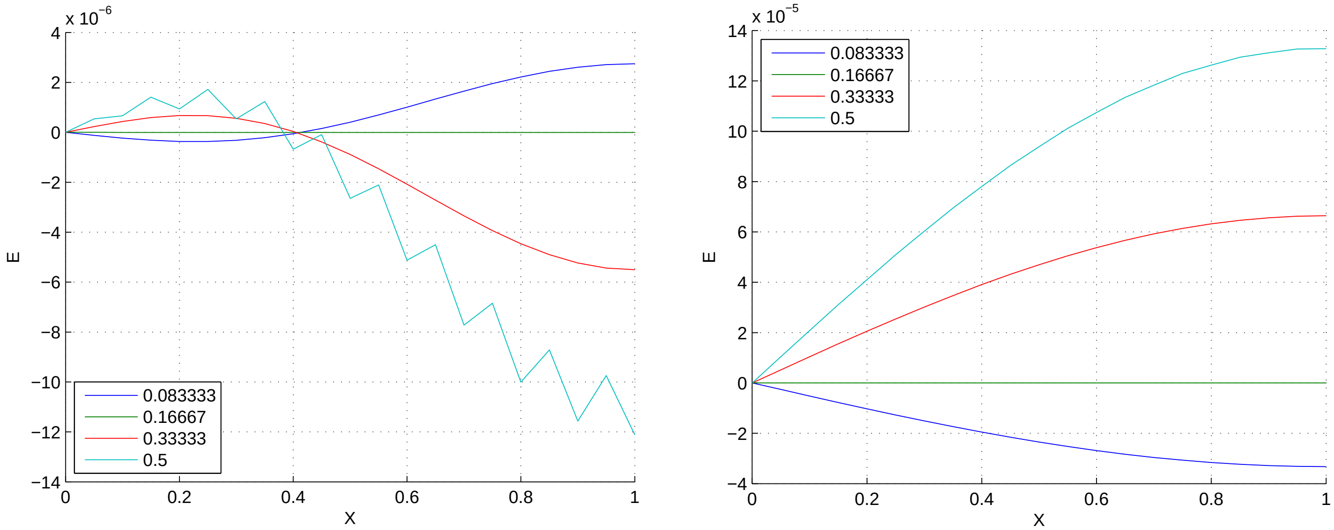
Across a sample of three points, there is for each Courant number a very precise linear correlation[6] whose gradient is always (very close to) 4 in the case $C = \frac{1}{6}$ and 2 in the remaining cases

[3] As before, exact division ensures the fidelity of the output for the suggested values. The range of T-values can in fact be extended to include multiples of 0.08 while maintaining exactness.

[4] As distinct from the absolute *value of the absolute* error.

[5] This claim is easy to check but, for the interested, evidence is provided in the supplementary image P3a.png.

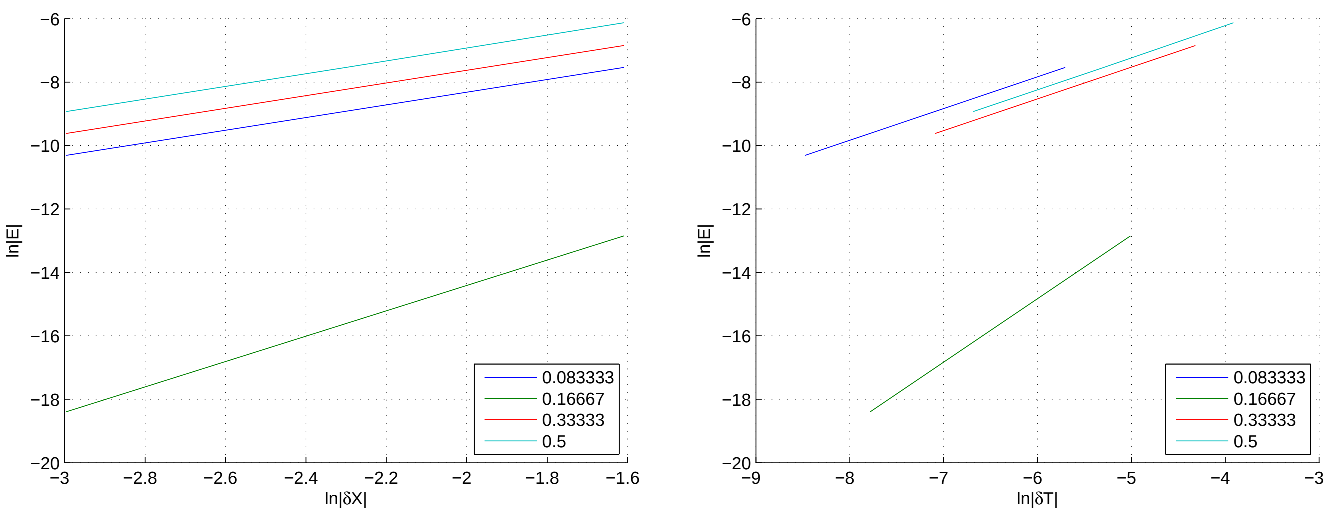
[6] Check this using, e.g., MATLAB's *Basic Fitting* plot tool.



(a) $T = 0.4$ (an atypical case).

(b) $T = 1.2$ (a typical case).

Figure 3: Various values of C with $N = 20$ and two choices of T.



(a) W.r.t. X.

(b) W.r.t. T.

Figure 4: Order of accuracy in both variables, with various valued of C sampled at $(X, T) = (1, 1.2)$.

$C = \frac{1}{12}, \frac{1}{3}, \frac{1}{2}$. A similar study with δT replacing δX for various values of C (with `Q3c`, again fixing (X, T) and varying N to attain a sample of values of δT) yields the same behaviour with the same degree of unanimity, this time with gradients 2 in the case $C = \frac{1}{6}$ and 1 in the remaining cases (see figure 4b).

Hence, it follows[7] with reasonable certainty[8] that the numerical scheme has orders of accuracy $O((\delta X)^4)$, $O((\delta T)^2)$ in the case $C = \frac{1}{6}$ and $O((\delta X)^2)$, $O(\delta T)$ in the other stable cases of Courant number. A striking consequence of this is that the X-order is always twice the T-order, paralleling

[7] Modelling with a linear regression, we have $\log(E) = m \log(\delta X) + c$, where m is the gradient, whence $|E| = e^c(\delta X)^m$, that is, $E = O((\delta X)^m)$. The accuracy of the scheme relative to X (T) is taken to be the order of the error E relative to δX (δT), fixing a point (X, T) and a change δT (δX).

[8] Each sample is small, consisting of three points, but is corroborated by the quality of the linear correlation and the unanimity across various values of (X, T).

the orders of their respective partial derivatives, and indeed that, excluding one resonant case, the orders are precisely those of the partial derivatives.

Q3: Theoretical stability and accuracy. This numerical scheme is known as the *Forward-Time Central-Space (FTCS)* finite-difference method, typically used to solve the diffusion equation and other parabolic PDEs. Theoretically, it is predicted that the scheme is stable iff the Courant number satisfies $C \leq \frac{1}{2}$, which is corroborated by the evidence. Furthermore, the general accuracy is indeed theoretically first-order in time and second-order in space.[9]

[9] See, for example, http://web.stanford.edu/~acolavin/files/finite_difference.pdf (section 2 for the general accuracy of finite-difference methods, section 4 for the stability with FTCS as a worked example; access date: 28/04/2015), for more information and derivations.

A Programs

The programs take the form of a module, `A.m`, which provides methods for the project, and five scripts, `Q*.m`, which use these methods to generate output for each question. These depend on two auxiliary, unlisted scripts that process output and are based on scripts from the internet:

`tabler(L,d,f)`: Appends the table at global variable T in LaTeX-format to the file L.pdf with heading d and formatting f, specified as a cell array of statements in MATLAB fprintf format.

`grapher(P)`: Writes the figure at global variable G in pdf-format to the file P.pdf.

A.1 Documentation

This section describes the purpose of the project's functions.

`A.F(x,t)`: Evaluates the semi-infinite solution at `(x,t)`.

`A.U1(x,t,n)`: Sums `n` terms of the finite fixed solution at `(x,t)`.

`A.U2(x,t,n)`: Sums `n` terms of the finite insulated solution at `(x,t)`.

`A.Y(N,C,S)`: Returns an $(\mathtt{N}+1) \times (\mathtt{S}+1)$ carpet generated by the numerical scheme (including initial and boundary values), with `N` X-steps, `S` T-steps and Courant number `C`.

A.2 A

```
classdef A
    methods(Static)
    function y = F(x,t)
        y = erfc(0.5*x/sqrt(t));
    end
    function y = U1(x,t,n)
        s = @(i) (2/(i*pi)) * exp(-t*(i*pi)^2) * sin(i*pi*x);
        y = 1-x - sum(arrayfun(s,1:n));
    end
    function y = U2(x,t,n)
        s = @(i) (2/((i-0.5)*pi)) * exp(-t*((i-0.5)*pi)^2) * sin
           ((i-0.5)*pi*x);
        y = 1 - sum(arrayfun(s,1:n));
    end
    function y = Y(N,C,S)
        y = zeros(N+1,S+1); y(1,:) = 1; y(:,1) = 0; y(1,1) = 0.5;
        for j = 2:S+1
            for i = 2:N
                y(i,j) = y(i,j-1) + C*(y(i+1,j-1) - 2*y(i,j-1) +
                   y(i-1,j-1));
            end
            y(N+1,j) = (1-2*C)*y(N+1,j-1) + 2*C*y(N,j-1);
        end
    end end
end
```

A.3 Q2

```
t = 0.2; x = (0:0.1:1)'; n = 400;
tu1 = @(r) A.U1(r,t,n); U1 = arrayfun(tu1,x);
tu2 = @(r) A.U2(r,t,n); U2 = arrayfun(tu2,x);
tf = @(r) A.F(r,t);      F = arrayfun(tf,x);

T = table(x, U1, U2, F)
tabler('L2','',{'%2.1f',1,'%8.7f',3})

for t = [0.05, 0.2, 0.4, 1, 1.6];
    tu1 = @(r) A.U1(r,t,n);
    tu2 = @(r) A.U2(r,t,n);
    tf = @(r) A.F(r,t);
    G = figure; clf; hold all; grid on
        fplot(tu1,0:1); fplot(tu2,0:1); fplot(tf,0:1);
        legend('Finite␣(fixed)','Finite␣(insulated)','Semi-
           Infinite','Location','SouthWest');
        grapher(strcat('P2-',num2str(t*100)))
end
```

A.4 Q3

```
N = 5; C = 1/2; n = 400;
dt = C/N^2;
frmt = {'%1.0f',1,'%8.7f',5};
[x,t] = meshgrid(0:1/N:1,0:0.4:2);

f = @(x,t) A.U2(x,t,n); Y = arrayfun(f,x,t)      % Exact
T = Y; tabler('L3a','y',frmt);

yr = A.Y(N,C,round(2/dt))'; y = [];              % Numerical
for i = 0:5
    y = [y;yr(1+i*round(0.4/dt),:)];
end
y
T = y; tabler('L3a','Y',frmt);

e = Y - y                                        % Error
T = e; tabler('L3a','e',frmt);
```

A.5 Q3a

```
N = 20; T = 0.4; n = 400;

G = figure; clf; hold all; grid on; l = {};
    for C = [1/12, 1/6, 1/3, 1/2]
        dt = C/N^2; S = T/dt; S = round(S);

        x = 0:1/N:1;
        u = @(r) A.U2(r,T,n); y = arrayfun(u,x);
        Y = A.Y(N,C,S); Y = Y(:,S+1)';
        e = (Y - y);
        plot(x,e)
        l = [l, num2str(C)];
    end
    legend(l,'Location','southwest'); ylabel('E'); xlabel('X')
grapher('P3aii')
```

A.6 Q3b

```
X = 1; T = 1.2; n = 400;
U = A.U2(X,T,n);
dx = 1./[5, 10, 20];

G = figure; clf; hold all; grid on; l = {};
    for C = [1/12,1/6,1/3,1/2]
        E = [];
        for N = [5, 10, 20]
            dt = C/N^2; S = T/dt; S = round(S);
            B = A.Y(N,C,S);
            Y = B(1+round(X*N),end);
            e = Y - U;
            E = [E,e];
        end
        plot(log(dx),log(abs(E)))
        l = [l, num2str(C)];
    end
    legend(l,'Location','southeast'); ylabel('ln|E|'); xlabel('ln
       |\deltaX|')
grapher('P3b')
```

`Q3c` is a slight modification of `Q3b`, converting a sample parametrised by N into a sample of δT, this time depending on C.